

Semi-Helices of Euclidean Spaces

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ABSTRACT. We introduce semi-helix hyper surfaces of Euclidean spaces. We also provide a local characterization of how these semi-helices are constructed.

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1. INTRODUCTION

A helix manifold is defined by the property that the angle between the tangent space at each point with a fix direction, in the ambient euclidean space, is constant. The importance of helices comes from their vast application in nature, sciences and also engineering of mechanical tools. They arise in the field of computer aided design, computer graphics, simulation of kinematic motion

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or design of highways, the shape of DNA and carbon nanotubes [1]. Helical structures often appear in fractal geometry [8].

The application of constant angle hyper-surfaces in physics of liquid crystal has been studied in [7] by Cermelli and Scala. In [6] it is observed that shadow boundaries are related to helix submanifolds. Also in [4] and [5], helix surfaces have been studied in non flat ambient spaces. The authors in [2] have proceeded with "helix submanifolds" i.e. submanifolds with constant angle between their tangent spaces and a fixed direction d .

In this paper we generalize the concept of '*Helix*', in the sense that the angle between tangent spaces and a fixed direction d can vary in a given sufficiently small interval $(\theta_0 - \varepsilon, \theta_0 + \varepsilon)$.

The paper is organized as follows: Section 2 contains definition of semi-helix hypersurfaces and in section 3 we present a full description of how this semi-helices are constructed. Finally, in the last part we show that all helices are locally constructed in the same way.

Throughout this paper, we follow Scala and Ruiz [2]. As is customary, we consider $\langle \cdot, \cdot \rangle$ as an inner product on \mathbb{R}^n and $T_p M$ is the tangent space at a point p on the hypersurface M in \mathbb{R}^n .

2. Semi-helix hypersurfaces

Let $d \in \mathbb{R}^n$ be any direction (i.e. a unitary vector) and let $V \subset \mathbb{R}^n$ be a linear subspace. The angle θ between d and V is the angle between the vectors d and $\pi_V(d)$, where $\pi_V : \mathbb{R}^n \rightarrow V$ is defined to be the orthogonal projection onto V . That is to say $\cos(\theta) := \langle d, \pi_V(d) \rangle$.

The definition of *semi-helices* is given here:

Definition 2.1. Let $M \subset \mathbb{R}^n$ be a hypersurface and $d \neq 0$ be a unitary fixed vector in \mathbb{R}^n . We say that M is a *semi-helix w.r. to d* and an initially chosen angle θ_0 , if there exists a fixed real number $0 \leq \varepsilon < \frac{\pi}{2}$, such that for all $p \in M$, the angle between d and $T_p M$, which we denote it by $\theta := \theta(p)$, belongs to the interval $(\theta_0 - \varepsilon, \theta_0 + \varepsilon)$.

The semi-helix submanifold M with domain of angle $(\theta_0 - \varepsilon, \theta_0 + \varepsilon)$ will be denoted by $M_\varepsilon(\theta_0)$.

Note that if $\varepsilon = 0$ then $M_0(\theta_0) = M(\theta_0)$ is a helix (manifold) of angle θ_0 .

Remark 2.2. Note that the angle θ comes from the angle between two "directions": d and the normal to the tangent space at each point. Hence if the spin of θ is in the direct sense, θ is considered to be *positive* and otherwise θ is *negative*.

Remark 2.3. Suppose M , θ_0 and d are the same as in definition 2.1 and also assume that $\xi : M \rightarrow \nu(M)$ is a unit normal vector field, where $\nu(M)$ denotes the normal vector bundle. Then we can use $\langle d, \xi_p \rangle = \sin(\theta_p)$ rather than $\cos(\theta_p)$ to identify the domain of angle θ_p .

Now we give a method to construct semi-helices in \mathbb{R}^n . Without loss of generality we can assume that θ_0 is *zero*.

3. Construction

First we choose fixed $0 < \varepsilon < \frac{\pi}{2}$, and consider $(-\varepsilon, \varepsilon)$ the range of θ . Let $H \subset \mathbb{R}^{n-1}$ be a hypersurface in \mathbb{R}^{n-1} and η be a unitary normal vector field on H . We can immerse H in \mathbb{R}^n in a canonical way as $(x_1, x_2, \dots, x_{n-1}) \rightarrow (x_1, x_2, \dots, x_{n-1}, 0)$.

without losing generality, we assume that $d = (0, \dots, 0, 1) \in \mathbb{R}^n$. put $M := H \times S^1(r)$, in which $S^1(r)$ is the circle of radius r . Now we define vector fields T_θ and T_ϕ as follows:

$$T_\theta(x) := \sin(\theta)\eta(x) + \cos(\theta)d, \quad (3.1)$$

$$T_\phi(x) := \cos(\phi)\eta(x) + \sin(\phi)d, \quad (3.2)$$

where $x \in H$ and $\phi = \frac{\pi - \theta}{2}$. See figure 1

Now we define the immersion $f_\varepsilon : H \times S^1(r) \rightarrow \mathbb{R}^n$ as follows:

$$(x, (r \cos(\phi), r \sin(\phi))) \mapsto x + r\sqrt{(2(1 - \cos \theta))}T_\phi(x)$$

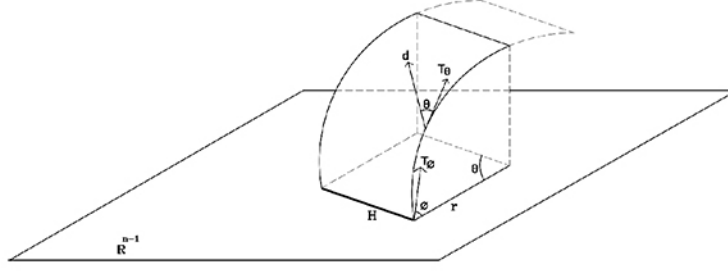


FIGURE 1

Note that $-\varepsilon < \theta < \varepsilon$. Therefore for enough small ε , f_ε is an immersion.

Theorem 3.1. *The immersed submanifold $M_\varepsilon = f_\varepsilon(M)$ is a semi-helix.*

Proof. First we will find a unitary normal vector field ξ_θ of the M_ε and show that its angle with d range in the interval $(-\varepsilon, \varepsilon)$. In order to do this we note that $f_\varepsilon|_H = Id_H$ and we can identify H with $f_\varepsilon\{H \times (r, 0)\}$ and M with M_ε . Thus the tangent space of M_ε at each point $f_\varepsilon(x, Y)$ is given by:

$$T_{(x,Y)}M_\varepsilon := T_xH \oplus \mathbb{R}T_\theta(x), \text{ for any } x \in H \text{ and } Y \in S^1(r).$$

Now let $\xi_\theta(x) := -\cos(\theta)\eta(x) + \sin(\theta)d$. Since η and d are orthogonal to H and $\langle \xi_\theta(x), T_\theta(x) \rangle = 0$ thus $\xi_\theta(x)$ is a unitary normal vector field on M_ε . Now orthogonality of η and d imply that $\langle \xi_\theta, d \rangle = \sin(\theta)$ and this complete the proof. \square

4. Reconstruction

In this section we show that each *semi-helix* $M_\varepsilon \subset \mathbb{R}^n$, can be obtained locally in the form that described in the previous section. Suppose that ξ_θ and T_θ are the unitary normal and unitary tangent vector fields respectively on *semi-helix* M_ε such that:

$$d = \cos(\theta)T_\theta(p) + \sin(\theta)\xi_\theta(p)$$

Lemma 4.1. *The integral curves of T_θ are sections of an $S^1(r)$ bundle over M .*

Proof. Let $p \in M_\varepsilon$. by definition, in this point $\cos(\theta) \neq 0$. Now let Q_p be a hyperplane that contains p and is orthogonal to d . Put $H := Q_p \cap M_\varepsilon$ which is a submanifold of M_ε . There is a vector field η which is normal to H and satisfies: $T_\theta(p) = \sin(\theta)\eta(p) + \cos(\theta)d$.

Now assume that $\alpha : (t_1, t_2) \longrightarrow M_\varepsilon$ is an integral curve of vector field T_θ . So for $t_p \in (t_1, t_2)$ we have $\alpha(t_p) = p$ and

$$\alpha'(t) = T_\theta(\alpha(t)) = \sin(\theta)\eta(\alpha(t)) + \cos(\theta)d, \quad \forall t \in (t_1, t_2)$$

Since η and d are orthogonal, for all t for which the angle θ corresponding with $\alpha(t) \in M_\varepsilon$ is positive, put:

$$e_1 := -\eta(\alpha(t)) \quad , \quad e_2 := d$$

and for other t , put

$$e_1 := \eta(\alpha(t)) \quad , \quad e_2 := d$$

Now we consider the extended basis $\beta = \{e_1, e_2, \dots, e_n\}$ of $T_{\alpha(t)}\mathbb{R}^n$. For $\alpha'(t) \in T_{\alpha(t)}\mathbb{R}^n$ we have:

$$\alpha'(t) = -\sin(\theta)e_1 + \cos(\theta)e_2 \tag{4.1}$$

On the other hand consider the trivial chart (Id, \mathbb{R}^n) around $\alpha(t)$, such that β is a basis for $T_{\alpha(t)}\mathbb{R}^n$ w.r. to this chart. Let $\alpha(t) = (\alpha_1(t), \dots, \alpha_n(t))$ be the local representation of α w.r. to chart (Id, \mathbb{R}^n) . Thus we have:

$$\alpha'(t) = \alpha'_1(t)e_1 + \dots + \alpha'_n(t)e_n. \tag{4.2}$$

equations 4.1 and 4.2 imply:

$$\alpha'_1(t) = -\sin(\theta), \quad \alpha'_2(t) = \cos(\theta), \quad \alpha'_i(t) = 0, \quad (i = 3, \dots, n)$$

Note that it is easy to show that, there is a linear relation between θ and t of the form $\theta(t) = at + c$, where $a, c \in \mathbb{R}$ are constant. Therefor we have:

$$\begin{aligned} \alpha'_1(t) &= -\sin(at + c), \\ \alpha'_2(t) &= \cos(at + c), \end{aligned}$$

$$\alpha_i'(t) = 0, \quad (i = 3, \dots, n).$$

Thus:

$$\alpha_1(t) = \frac{1}{a} \cos(at + c) + c_1,$$

$$\alpha_2(t) = \frac{1}{a} \sin(at + c) + c_2,$$

$$\alpha_i(t) = c_i \quad (i = 3, \dots, n).$$

where c_i 's are constant. By using initial condition $\alpha(t_p) = p$, we can obtain the values of c_i 's. The last relation shows that α (the integral curve of T_θ) is a section of a trivial $S^1(r)$ bundle on M with $r = \frac{1}{a}$. \square

Theorem 4.2. *Each semi-helix hypersurface is locally isomorphic to the semi-helix hypersurface whose construction was explained in section 3.*

Proof. Let M_ε be a semi-helix hypersurface w.r. to a fixed direction d , and T_θ be the tangent component of d along M_ε as mentioned in lemma 4.1.

We choose an arbitrary $p \in M_\varepsilon$ and let Q_p be a hyperplane passing through p and orthogonal to d . Consider the submanifold $H := Q_p \cap M_\varepsilon$ of M_ε . It is obvious that from each point $x \in H$, passes a curve α with values in $S^1(r)$ such that $T_\theta(x) = \alpha'(t_x)$, ($\alpha(t_x) = x$).

We denote the integral curve of T_θ passing through p by $\alpha_p : I \rightarrow M_\varepsilon$. In fact for all $t \in I$ the angle between $\alpha'(t)$ and d is θ_t , that varies in the interval $(\varepsilon_1, \varepsilon_2)$.

Now we extend the curve α_p to $\beta_p : J \rightarrow \mathbb{R}^n$ such that β_p is a section of $S^1(r)$ bundle, $I \subset J$ and $\beta_p|_I = \alpha_p$. Also there exist a point $q = \beta_p(t_q) \in \beta_p(J)$ such that the angle between d and $\beta_p'(t_q)$ is zero.

Finally, we consider the hyperplane Q_q , that contains q and is orthogonal to d . In fact Q_q is a translation of Q_p and q is a translation of p under the same translation.

We denote the translated submanifold H of Q_p , by H' . Let θ_p be the angle between d and $T_\theta(p)$, then:

$$p = q + r\sqrt{2(1 - \cos(\theta_p))}T_{\phi_p}(q)$$

where $\phi_p = \frac{\pi - \theta_p}{2}$

Now if $U \subset M_\varepsilon$ is a neighborhood around p , for all $y \in U \cap H$ we have:

$$y = y' + r\sqrt{2(1 - \cos(\theta_y))}T_{\phi_y}(y')$$

where $\phi_y = \frac{\pi - \theta_y}{2}$ and $y' \in H'$ is a translation of y . This indicates that $U \subset f_\varepsilon(H' \times S^1(r))$ and $f_\varepsilon(H' \times S^1(r))$ is, as we showed in the previous section, a semi-helix hypersurface. This completes the proof. □

5. Conclusion

Our description indicates that semi-helix hypermanifolds of a euclidean space are locally isomorphic to the ones constructed in "Section 3", even though their global structure might involve some subtleties that could make them not to be isomorphic to our manifolds.

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